LETTERPLACE AND CO-LETTERPLACE IDEALS OF POSETS

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Abstract. To a natural number n, a finite partially ordered set P and a poset ideal $\mathcal{J}$ in the poset $\mathrm{Hom}(P,[n])$ of isotonian maps from P to the chain on n elements, we associate two monomial ideals, the letterplace ideal $L(n,P;\mathcal{J})$ and the co-letterplace ideal $L(P,n;\mathcal{J})$. These ideals give a unified understanding of a number of ideals studied in monomial ideal theory in recent years. By cutting down these ideals by regular sequences of variable differences we obtain: multichain ideals and generalized Hibi type ideals, initial ideals of determinantal ideals, strongly stable ideals, d-partite d-uniform ideals, Ferrers ideals, edge ideals of cointerval d-hypergraphs, and uniform face ideals.

Introduction

Monomial ideals arise as initial ideals of polynomial ideals. For natural classes of polynomial ideals, like determinantal ideals of generic matrices, of generic symmetric matrices, and of Pfaffians of skew-symmetric matrices, their Gröbner bases and initial ideals [36], [26], [5] have been computed. In full generality one has the class of Borel-fixed ideals (in characteristic 0 these are the strongly stable ideals), which arise as initial ideals of any polynomial ideal after a generic change of coordinates. This is a very significant class in computational algebra [2], [19].

Monomial ideals have also grown into an active research area in itself. In particular one is interested in their resolution, and in properties like shellability (implying Cohen-Macaulayness), and linear quotients (implying linear resolution). Classes that in particular have been studied are generalized Hibi ideals [14], d-uniform hypergraph ideals [33], Ferrers ideals [7], [33], uniform face ideals [6] and [20], and cointerval d-hypergraph ideals [10].

This article presents a unifying framwork for these seemingly varied classes of monomial ideals by introducing the classes of letterplace and co-letterplace ideals associated to a a natural number n, a finite partially ordered set P, and a poset ideal $\mathcal{J}$ in the poset $\mathrm{Hom}(P,[n])$ of isotone maps $P \to [n]$, where $[n]$ is the chain on n elements. Many basic results of the abovementioned articles follow from the general results we give here.

As it turns out, most of the abovementioned classes of ideals are not letterplace and co-letterplace ideals. Rather they *derive* from these ideals by dividing out by a *regular sequence* of variable differences. The main technical results of the present paper is to give criteria for when such a sequence of variable differences is regular, Theorems 2.1, 2.2, 5.6, and 5.12. Thus we get classes of ideals, with the same homological properties as letterplace and co-letterplace ideals. Also we show

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that letterplace and co-letterplace ideals in themselves may not be derived from other monomial ideals by cutting down by a regular sequence of varibale differences, Lemma 2.8. These ideals therefore have the flavor of beeing "free" objects in the class of monomial ideals. We see this as accounting for many of their nice properties, see [8] and [16].

In [14] V.Ene, F.Mohammadi, and the third author introduced the classes of generalized Hibi ideals, associated to natural numbers $s \leq n$ and a finite partially ordered set P, and investigated these ideals. This article both generalizes this, and provides a hightened understanding of these ideals. There is an extra parameter s involved here, but we show that these ideals can be understood as the letterplace ideal associated to the natural number n and the partially ordered set $P \times [n-s+1]$, after we divide this ideal out by a regular sequence of variable differences.

The *n'th letterplace ideal* associated to n and P is the monomial ideal, written $L(n,P)$, generated by all monomials

$$x_{1,p_1}x_{2,p_2}\cdots x_{n,p_n}$$

where $p_1 \leq p_2 \leq \cdots \leq p_n$ is a multichain in P. These ideals $L(n,P)$ were shown in [14] to be Cohen-Macaulay (CM) ideals (actually shellable) of codimension equal to the cardinality of P. These ideals are all generated in degree n. If P is the antichain on d elements, then $L(n,P)$ is a complete intersection, and defines a quotient ring of mulitplicity n^d. If P is the chain on d elements, then $L(n,P)$ is the initial ideal of the ideal of maximal minors of a generic $n \times (n+d-1)$ matrix. The quotient ring has multiplicity $\binom{n+d-1}{d}$. These are respectively the maximal and minimal multiplicity of CM ideals of codimension d generated in degree n. As P varies among posets of cardinality d we therefore get ideals interpolating between these extremal cases.

Its Alexander dual, the *n'th co-letterplace ideal* associated to n and P, written $L(P,n)$, is the ideal generated by all monomials

$$\prod_{p\in P} x_{p,i_p}$$

where $p < q$ in P implies $i_p \leq i_q$. This is an ideal with linear quotients [14], and therefore linear resolution.

The Cohen-Macaulay ideals $L(n,P)$ are all generated in a single degree n. To obtain CM ideals with varying degrees on generators, we now add an extra layer of structure. Given a poset ideal $\mathcal{J}$ in the poset $\mathrm{Hom}(P,[n])$ of isotone maps $P \to [n]$, we get first more generally the co-letterplace subideal $L(P,n;\mathcal{J}) \subseteq L(P,n)$. This ideal also has linear quotients, Theorem 5.1. When P is a chain on d elements, all strongly stable ideals generated in degree d are regular quotients of these co-letterplace ideals, Example 6.1. As P varies we therefore get a substantial generalization of the class of strongly stable ideals generated in a single degree, and with much the same nice homological behaviour.

The Alexander dual of $L(P,n;\mathcal{J})$ we denote by $L(n,P;\mathcal{J})$, and is explicitly described in Theorem 5.9. Since the former ideal has linear resolutions, by [11] the latter ideal is Cohen-Macaulay and it contains the letterplace ideal $L(n,P)$. Even when P is the chain on d elements, all h-vectors of embedding dimension d of graded Cohen-Macaulay ideals (in a polynomial ring), may be realized for such ideals $L(P,n;\mathcal{J})$. This is therefore a very large class of Cohen-Macaulay ideals.

Dividing out a monomial ring S/I by an difference of variables $x_a - x_b$, corresponds to setting the variables x_a and x_b equal in I to obtain a new monomial ideal J. In this article we therefore naturally introduce the notion of separation of variables, Definition 2.7, of a monomial ideals: I is obtained from J by a *separation* of variables, if $x_a - x_b$ is a regular element for S/I. Surprisingly this simple and natural notion does not seem to have been a topic of study in itself before for monomial ideals, but see [15]. In particular the behaviour of Alexander duality when dividing out by such a regular sequence of variable differences, is given in Proposition 7.2, and Theorems 2.10 and 7.3.

After the appearance of this article as a preprint, a number of further investigations has been done on letterplace and co-letterplace ideals. The article [27] studies more generally ideals $L(P, Q)$ where both P and Q are finite partially orded sets, and [23] investigates Alexander duality of such ideals. In [9] resolutions of letterplace ideals $L(n, P)$ are studied, in particular their multigraded Betti numbers are computed. [8] gives explicitly the linear resolutions of co-letterplace ideal $L(P, n; \mathcal{J})$, thereby generalizing the Eliahou-Kervaire resolution for strongly stable ideals generated in a single degree. It computes the canonical modules of the Stanley-Reisner rings of letterplace ideals $L(n, P; \mathcal{J})$. They have the surprising property of being multigraded ideals in these Stanley-Reisner rings. A related and remarkable consequence is that the simplicial complexes associated to letterplace ideals $L(n, P; \mathcal{J})$ are triangulations of balls. Their boundaries are therefore triangulations of spheres, and this class of sphere triangulations comprehensively generalizes the class of Bier spheres [3]. The notion of separation is further investigated in [24] and in [1], which shows that separation corresponds to a deformation of the monomial ideal, and identifies the deformation directions in the cotangent cohomology it corresponds to. In [16] deformations of letterplace ideals $L(2, P)$ are computed when the Hasse diagram has the structure of a rooted tree. The situation is remarkably nice. These ideals are unobstruced, and the full deformation family can be explicitly computed. This deformed family has a polynomial ring as a base ring, and the ideal of the full family is a rigid ideal. In some simple example cases these are the ideals of 2-minors of a generic $2 \times n$ matrix, and the ideal of Pfaffians of a generic skew-symmetric 5×5 matrix.

The organization of the paper is as follows. In Section 1 we define ideals $L(Q, P)$ associated to pairs of posets Q and P. In particular for the totally ordered poset $[n]$ on n elements, we introduce the letterplace ideals $L([n], P)$ and co-letterplace ideals $L(P, [n])$. We investigate how they behave under Alexander duality. In Section 2 we study when a sequence of variable differences is regular for letterplace and co-letterplace ideals. We also define the notion of separation. Section 3 gives classes of ideals, including generalized Hibi ideals and initial ideals of determinantal ideals, which are quotients of letterplace ideals by a regular sequence. Section 4 describes in more detail the generators and facets of various letterplace and co-letterplace ideals. Section 5 considers poset ideals $\mathcal{J}$ in $\mathrm{Hom}(P, [n])$ and the associated co-letterplace ideal $L(P, n; \mathcal{J})$. We show it has linear resolution, and compute its Alexander dual $L(n, P; \mathcal{J})$. Section 6 gives classes of ideals which are quotients of co-letterplace ideals by a regular sequence. This includes strongly stable ideals, d-uniform d-partite hypergraph ideals, Ferrers ideals, and uniform face ideals. The last sections 7 and 8 contain proofs of basic results in this paper on when sequences of variable

differences are regular, and how Alexander duality behaves when cutting down by such a regular sequence.

1. Letterplace ideals and their Alexander duals

If P is a partially ordered set (poset), a *poset ideal* $J \subseteq P$ is a subset of P such that $q \in J$ and $p \in P$ with $p \leq q$, implies $p \in J$. The term *order ideal* is also much used in the literature for this notion. If S is a subset of P, the poset ideal generated by S is the set of all elements $p \in P$ such that $p \leq s$ for some $s \in S$.

1.1. **Isotone maps.** Let P and Q be two partially ordered sets. A map $\phi : Q \to P$ is *isotone* or *order preserving*, if $q \leq q'$ implies $\phi(q) \leq \phi(q')$. The set of isotone maps is denoted $\mathrm{Hom}(Q, P)$. It is actually again a partially ordered set with $\phi \leq \psi$ if $\phi(q) \leq \psi(q)$ for all $q \in Q$. The following will be useful.

Lemma 1.1. *If P is a finite partially ordered set with a unique maximal or minimal element, then an isotone map $\phi : P \to P$ has a fix point.*

Proof. We show this in case P has a unique minimal element $p = p_0$. Then $p_1 = \phi(p_0)$ is $\geq p_0$. If $p_1 > p_0$, let $p_2 = \phi(p_1) \geq \phi(p_0) = p_1$. If $p_2 > p_1$ we continue. Since P is finite, at some stage $p_n = p_{n-1}$ and since $p_n = \phi(p_{n-1})$, the element p_{n-1} is a fix point. □

1.2. **Alexander duality.** Let $\Bbbk$ be a field. If R is a set, denote by $\Bbbk[x_R]$ the polynomial ring in the variables x_r where r ranges over R. If A is a subset of R denote by m_A the monomial $\Pi_{a \in A} x_a$.

Let I be a squarefree monomial ideal in a polynomial ring $\Bbbk[x_R]$, i.e. its generators are monomials of the type m_A. It corresponds to a simplicial complex Δ on the vertex set R, consisting of all $S \subseteq R$, called faces of Δ, such that $m_S \notin I$.

The Alexander dual J of I, written $J = I^A$, may be defined in different ways. Three definitions are the following.
1. The Alexander dual J is the monomial ideal in $\Bbbk[x_R]$ whose monomials are those with nontrivial common divisor with every monomial in I.
2. The Alexander dual J is the ideal generated by all monomials m_S where the S are complements in R of faces of Δ.
3. If $I = \cap_{i=1}^r \mathfrak{p}_r$ is a decomposition into prime monomial ideals $\mathfrak{p}_i$ where $\mathfrak{p}_i$ is generated by the variables x_a as a ranges over the subset A_i of R, then J is the ideal generated by the monomials m_{A_i}, $i = 1, \ldots, r$. (If the decomposition is a minimal primary decomposition, the m_{A_i} is a minimal generating set of J.)

1.3. **Ideals from Hom-posets.** To an isotone map $\phi : Q \to P$ we associate its graph $\Gamma\phi \subseteq Q \times P$ where $\Gamma\phi = \{(q, \phi(q)) \mid q \in Q\}$. As ϕ ranges over $\mathrm{Hom}(Q, P)$, the monomials $m_{\Gamma\phi}$ generate a monomial ideal in $\Bbbk[x_{Q\times P}]$ which we denote by $L(Q, P)$. More generally, if $\mathcal{S}$ is a subset of $\mathrm{Hom}(Q, P)$ we get ideals $L(\mathcal{S})$ generated by $m_{\Gamma\phi}$ where $\phi \in \mathcal{S}$.

If R is a subset of the product $Q \times P$, we denote by R^τ the subset of $P \times Q$ we get by switching coordinates. As $L(Q, P)$ is an ideal in $\Bbbk[x_{Q\times P}]$, we may also consider it as an ideal in $\Bbbk[x_{P\times Q}]$. In cases where we need to be precise about this, we write it then as $L(Q, P)^\tau$.

If Q is the totally ordered poset on n elements $Q = [n] = \{1 < 2 < \cdots < n\}$, we call $L([n], P)$, written simply $L(n, P)$, the n'th *letterplace ideal* of P. It is generated by the monomials

$$x_{1,p_1} x_{2,p_2} \cdots x_{n,p_n} \text{ with } p_1 \leq p_2 \leq \cdots \leq p_n.$$

This is precisely the same ideal as the multichain ideal $I_{n,n}(P)$ defined in [14] (but with indices switched). The ideal $L(P, [n])$, written simply $L(P, n)$, is the n'th *co-letterplace ideal* of P. In [14] it is denoted $H_n(P)$ and is called a generalized Hibi type ideal. For some background on the letterplace notion, see Remark 1.7 at the end of this section.

The following is Theorem 1.1(a) in [14], suitably reformulated. Since it is a very basic fact, we include a proof of it.

Proposition 1.2. *The ideals $L(n, P)$ and $L(P, n)^\tau$ are Alexander dual in $\Bbbk[x_{[n]\times P}]$.*

Proof. Let $L(n, P)^A$ be the Alexander dual of $L(n, P)$. First we show $L(P, n) \subseteq L(n, P)^A$. This is equivalent to: For any $\phi \in \mathrm{Hom}([n], P)$ and any $\psi \in \mathrm{Hom}(P, [n])$, the graphs $\Gamma\phi$ and $\Gamma\psi^\tau$ intersect in $[n] \times P$. Let i be a fix point for $\psi \circ \phi$. Then $i \xrightarrow{\phi} p \xrightarrow{\psi} i$ and so (i, p) is in both $\Gamma\phi$ and $\Gamma\psi^\tau$.

Secondly, given a squarefree monomial m in $L(n, P)^A$ we show that it is divisible by a monomial in $L(P, n)$. This will show that $L(n, P)^A \subseteq L(P, n)$ and force equality here. So let the monomial m correspond to the subset F of $P \times [n]$. It intersects all graphs $\Gamma\phi^\tau$ where $\phi \in \mathrm{Hom}([n], P)$. We must show it contains a graph $\Gamma\psi$ where $\psi \in \mathrm{Hom}(P, [n])$. Given F, let $\mathcal{J}_n = P$ and let $\mathcal{J}_{n-1}$ be the poset ideal of P generated by all $p \in \mathcal{J}_n = P$ such that $(p, n) \notin F$. Inductively let $\mathcal{J}_{i-1}$ be the poset ideal in $\mathcal{J}_i$ generated by all p in $\mathcal{J}_i$ with (p, i) not in F.

Claim 1. $\mathcal{J}_0 = \emptyset$.

Proof. Otherwise let $p \in \mathcal{J}_0$. Then there is $p \leq p_1$ with $p_1 \in \mathcal{J}_1$ and $(p_1, 1) \notin F$. Since $p_1 \in \mathcal{J}_1$ there is $p_1 \leq p_2$ with $p_2 \in \mathcal{J}_2$ such that $(p_2, 2) \notin F$. We may continue this and get a chain $p_1 \leq p_2 \leq \cdots \leq p_n$ with (p_i, i) not in F. But this contradicts F intersecting all graphs $\Gamma\phi$ where $\phi \in \mathrm{Hom}([n], P)$. □

We thus get a filtration of poset ideals

$$\emptyset = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_{n-1} \subseteq \mathcal{J}_n = P.$$

This filtration corresponds to an isotone map $\psi : P \to [n]$.

Claim 2. $\Gamma\psi$ is a subset of F.

Proof. Let $(p, i) \in \Gamma\psi$. Then $p \in \mathcal{J}_i \backslash \mathcal{J}_{i-1}$ and so $p \notin \mathcal{J}_{i-1}$. Thus $(p, i) \in F$.

□

Remark 1.3. The case $n = 2$ was shown in [21] where the ideal H_P generated by $\prod_{p\in J} x_p \prod_{q\in P\backslash J} y_q$ as J varies over the poset ideals in P, was shown to be Alexander dual to the ideal generated by $x_p y_q$ where $p \leq q$. In [21] it is also shown that the $L(2, P)$ are precisely the edge ideals of bipartite Cohen-Macaulay graphs.

Remark 1.4. That $L(m, n)$ and $L(n, m)$ are Alexander dual is Proposition 4.5 of [17]. There the elements of these ideals are interpreted as paths in a $m \times n$ matrix with generic linear forms (x_{ij}) and the generators of the ideals are the products of the variables in these paths.

1.4. **Alexander dual of** $L(Q,P)$**.** In general $L(Q,P)$ and $L(P,Q)$ are not Alexander dual. This is easily checked if for instance Q and P are antichains of sizes ≥ 2. However we have the following.

Proposition 1.5. *Suppose Q has a unique maximal or minimal element. The least degree of a generator of the Alexander dual $L(Q,P)^A$ and of $L(P,Q)$ are both $d = |P|$ and the degree d parts of these ideals are equal. In particular, since $L(P,Q)$ is generated in this degree d, it is contained in $L(Q,P)^A$.*

Note that the above is equivalent to say that the minimal primes of $L(Q,P)$ of height $\leq |P|$ are precisely the

$$\mathfrak{p}_\psi = (\{x_{\psi(p),p} \ \ p \in P\}), \quad \text{where } \psi \in \mathrm{Hom}(P,Q).$$

Proof. We show that:

1. $L(Q,P) \subset \mathfrak{p}_\psi$ for all $\psi \in \mathrm{Hom}(P,Q)$.
2. $\mathfrak{p}_\psi$ is a minimal prime of $L(Q,P)$.
3. Any minimal prime $\mathfrak{p}$ of $L(Q,P)$ is $\mathfrak{p} = \mathfrak{p}_\psi$ for some ψ.

This will prove the proposition.
1. Given $\phi \in \mathrm{Hom}(Q,P)$ and $\psi \in \mathrm{Hom}(P,Q)$. We have to show that $m_\phi = \prod_{q\in Q} x_{q,\phi(q)} \in \mathfrak{p}_\psi$. By Lemma 1.1 $\psi \circ \phi$ has a fix point q, and let $p = \phi(q)$. Then $\psi(p) = q$. Therefore, $x_{q,p}$ is a factor of m_ϕ and a generator of $\mathfrak{p}_\psi$. This implies that $m_\phi \in \mathfrak{p}_\psi$.
2. Next we show that $\mathfrak{p}_\psi$ is a minimal prime ideal of $L(Q,P)$. Suppose this is not the case. Then we may skip one of its generators, say $x_{\psi(p),p}$, to obtain the prime ideal $\mathfrak{p} \subset \mathfrak{p}_\psi$ with $L(Q,P) \subset \mathfrak{p}$. Let $\phi \in \mathrm{Hom}(Q,P)$ be the constant isotone map with $\phi(q) = p$ for all $q \in Q$. Then $m_\phi = \prod_{q\in Q} x_{q,p} \in L(Q,P)$. Since no factor of m_ϕ is divisible by a generator of $\mathfrak{p}$, it follows that $L(Q,P)$ is not contained in $\mathfrak{p}$, a contradiction.
3. Now let $\mathfrak{p}$ be any minimal prime ideal of $L(Q,P)$. Since $L(Q,P) \subset \mathfrak{p}$ it follows as in the previous paragraph that for each $p \in P$ there exists an element $\psi(p) \in Q$ such that $x_{\psi(p),p} \in \mathfrak{p}$. This show that $\mathrm{height}\, L(Q,P) = |P|$. Assume now that $\mathrm{height}\, \mathfrak{p} = |P|$. Then $\mathfrak{p} = (\{x_{\psi(p),p} \ \ p \in P\})$. It remains to be shown that $\psi \ P \to Q$ is isotone. Suppose this is not the case. Then there exist $p, p' \in P$ such that $p < p'$ and $\psi(p) \not\leq \psi(p')$. Let $\phi : Q \to P$ the map with $\phi(q) = p$ if $q \leq \psi(p')$ and $\phi(q) = p'$ if $q \not\leq \psi(p')$. Then ϕ is isotone, and it follow that $m_\phi = \prod_{q \leq \psi(p')} x_{q,p} \prod_{q \not\leq \psi(p')} x_{q,p'}$ does not belong to $\mathfrak{p}$, a contradiction. □

Remark 1.6. In [23] they determine precisely when $L(P,Q)$ and $L(Q,P)$ are Alexander dual, for finite posets P and Q.

Remark 1.7. Let $X = \{x_1, \ldots, x_n\}$ be an alphabet. The letterplace correspondence is a way to encode non-commutative monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$ in $\Bbbk\langle X \rangle$ by commutative polynomials $x_{i_1,1} \cdots x_{i_r,r}$ in $\Bbbk[X \times \mathbb{N}]$. It is due to G.-C. Rota who again attributed it to a physicist, apparently Feynman. D. Buchsbaum has a survey article [4] on letterplace algebra, and the use of these technique in the resolution of Weyl modules. Recently [29] use letterplace ideals in computations of non-commutative Gröbner bases.

2. Quotients of letterplace ideals

A chain c in the product of two posets $Q \times P$ is said to be *left strict* if for two elements in the chain, $(q,p) < (q',p')$ implies $q < q'$. Analogously we define right strict. The chain is *bistrict* if it is both left and right strict.

An isotone map of posets $\phi : Q \times P \to R$ is said to have *left strict chain fibers* if all its fibers $\phi^{-1}(r)$ are left strict chains in $Q \times P^{op}$. Here P^{op} is the opposite poset of P, i.e. $p \leq^{op} p'$ in P^{op} iff $p' \leq p$ in P.

The map ϕ gives a map of linear spaces $\phi_1 : \langle x_{Q\times P}\rangle \to \langle x_R \rangle$ (the brackets here mean the $\Bbbk$-vector space spanned by the set of variables). The map ϕ_1 induces a map of polynomial rings $\hat{\phi} : \Bbbk[x_{Q\times P}] \to \Bbbk[x_R]$. In the following B denotes a basis for the kernel of the map of degree one forms ϕ_1, consisting of differences $x_{q,p} - x_{q',p'}$ with $\phi_1(q,p) = \phi_1(q',p')$.

Theorem 2.1. *Given an isotone map $\phi : [n] \times P \to R$ which has left strict chain fibers. Then the basis B is a regular sequence of the ring $\Bbbk[x_{[n]\times P}]/L(n,P)$.*

Theorem 2.2. *Given an isotone map $\psi : P \times [n] \to R$ which has left strict chain fibers. Then the basis B is a regular sequence of the ring $\Bbbk[x_{P\times[n]}]/L(P,n)$.*

We shall prove these in Section 8. For now we note that they require distinct proofs, with the proof of Theorem 2.2 the most delicate.

In the setting of Theorem 2.1, we let $L^\phi(n,P)$ be the ideal generated by the image of the n'th letterplace ideal $L(n,P)$ in $\Bbbk[x_R]$, and in the setting of Theorem 2.2, we let $L^\psi(P,n)$ be the ideal generated by the image of the n'th co-letterplace ideal $L(P,n)$ in $\Bbbk[x_R]$, Note that $L^\phi(n,P)$ is a squarefree ideal iff in the above the fibers $\phi^{-1}(r)$ are bistrict chains in $[n] \times P^{op}$, and similarly $L^\psi(P,n)$ is a squarefree ideal iff the fibers $\psi^{-1}(r)$ are bistrict chains in $P \times [n]^{op}$.

We get the following consequence of the above Theorems 2.1 and 2.2.

Corollary 2.3. *The quotient rings $\Bbbk[x_{[n]\times P}]/L(n,P)$ and $\Bbbk[x_R]/L^\phi(n,P)$ have the same graded Betti numbers. Similarly for $\Bbbk[x_{P\times[n]}]/L(P,n)$ and $\Bbbk[x_R]/L^\psi(P,n)$.*

Proof. We prove the first statement. Let $L^{\mathrm{im}\,\phi}(n,P)$ be the image of $L(n,P)$ in $\Bbbk[x_{\mathrm{im}\,\phi}]$, and $S = R \backslash \mathrm{im}\,\phi$. Thus $\Bbbk[x_{\mathrm{im}\,\phi}]/L^{\mathrm{im}\,\phi}(n,P)$ is a quotient of $\Bbbk[x_{[n]\times P}]/L(n,P)$ by a regular sequence, and $\Bbbk[x_R]/L^\phi(n,P)$ is $\Bbbk[x_{\mathrm{im}\,\phi}]/L^{\mathrm{im}\,\phi}(n,P) \otimes_\Bbbk \Bbbk[x_S]$. $\square$

For the poset P consider the multichain ideal $I(n,P)$ in $\Bbbk[x_P]$ generated by monomials $x_{p_1}x_{p_2}\cdots x_{p_n}$ where $p_1 \leq p_2 \leq \cdots \leq p_n$ is a multichain of length n in P. The quotient $\Bbbk[x_P]/I(n,P)$ is clearly artinian since x_p^n is in $I(n,P)$ for every $p \in P$.

Corollary 2.4. *The ring $\Bbbk[x_P]/I(n,P)$ is an artinian reduction of $\Bbbk[x_{[n]\times P}]/L(n,P)$ by a regular sequence. In particular $L(n,P)$ is a Cohen-Macaulay ideal. It is Gorenstein iff P is an antichain.*

Proof. The first part is because the map $[n] \times P \to P$ fulfills the criteria of Theorem 2.1 above. An artinian ideal is Gorenstein iff it is a complete intersection. Since all x_p^n are in $I(n,P)$, this holds iff there are no more generators of $I(n,P)$, which means precisely that P is an antichain. $\square$

This recovers part of Theorem 2.4 of [14] showing that $L(n,P)$ is Cohen-Macaulay. The Gorenstein case above is Corollary 2.5 of [14].

Remark 2.5. The multigraded Betti numbers of the resolution of $L(n,P)$ is described in [9], as well as other properties of this resolution.

Recall that a squarefree monomial ideal is bi-Cohen-Macaulay, [17], iff both the ideal and its Alexander dual are Cohen-Macaulay ideals.

Corollary 2.6. *$L(n,P)$ is bi-Cohen-Macaulay iff P is totally ordered.*

Proof. Since $L(n,P)$ is Cohen-Macaulay, it is bi-Cohen-Macaulay iff it has a linear resolution, [12]. Equivalently $I(n,P)$ in $\Bbbk[x_P]$ has a linear resolution. But since $I(n,P)$ gives an artinian quotient ring, this is equivalent to $I(n,P)$ being the n'th power of the maximal ideal. In this case every monomial $x_p^{n-1}x_q$ is in $I(n,P)$ and so every pair p,q in P is comparable. Thus P is totally ordered. Conversely, if P is totally ordered, then clearly $I(n,P)$ is the n'th power of the maximal ideal. □

Definition 2.7. Let $R' \to R$ be a surjective map of sets with R' of cardinality one more than R, and let $r_1 \neq r_2$ in R' map to the same element in R. Let I be a monomial ideal in $k[x_R]$. A monomial ideal J in $\Bbbk[x_{R'}]$ is a *separation* of I if i) I is the image of J by the natural map $\Bbbk[x_{R'}] \to \Bbbk[x_R]$, ii) x_{r_1} occurs in some minimal generator of J and similarly for x_{r_2}, and iii) $x_{r_1} - x_{r_2}$ is a regular element of $\Bbbk[x_{R'}]/J$.

The ideal I is *separable* if it has some separation J. Otherwise it is *inseparable*. If J is obtained from I by a succession of separations, we also call J a *separation* of I. We say that I is a *regular quotient by variable differences* of J, or simply a *regular quotient* of J. If J is *inseparable*, then J is a *separated model* for I.

This notion also occurs in [15] where inseparable monomial ideals are called maximal. The canonical example of a separation of a non-squarefree monomial ideal is of course its polarization.

Lemma 2.8. *Let I be an ideal generated by a subset of the generators of $L(Q,P)$. Then I is inseparable.*

Proof. Let $R' \to Q \times P$ be a surjective map with R' of cardinality one more than $Q \times P$. Suppose there is a monomial ideal J in $\Bbbk[x_{R'}]$ which is a separation of I. Let a and b in R' both map to (q,p). For any other element of R', we identify it with its image in $Q \times P$. Suppose $m = x_a m_0$ in J maps to a generator $x_{q,p}m_0$ of $L(Q,P)$, and $m' = x_b m_1$ maps to another generator of $L(Q,P)$. Then m_0 does not contain a variable $x_{q,p'}$ with first index q, and similarly for m_1. Note that the least common multiple m_{01} of m_0 and m_1 does not contain a variable with first index q. Hence m_{01} is not in $L(Q,P)$ and so m_{01} is not in J. But $(x_b - x_a)m_{01}$ is in J since $x_b m_{01}$ and $x_a m_{01}$ are in J. By the regularity of $x_b - x_a$ this implies m_{01} in J, a contradiction. □

As we shall see, many naturally occurring monomial ideals are separable and have separated models which are letterplace ideals $L(n,P)$ or are generated by a subset of the generators of co-letterplace ideals $L(P,n)$.

Remark 2.9. In [15, Section 2] the first author shows that the separated models of the squarefree power $(x_1,\ldots,x_n)_{sq}^{n-1}$ are in bijection with trees on n vertices.

We now consider the Alexander dual of $L^{\phi}(n,P)$.

Theorem 2.10. *Let $\phi : [n] \times P \to R$ be an isotone map such that the fibers $\phi^{-1}(r)$ are bistrict chains in $[n] \times P^{op}$. Then the ideals $L^{\phi}(n, P)$ and $L^{\phi^{\tau}}(P, n)$ are Alexander dual.*

We prove this in Section 7.

Remark 2.11. The Alexander dual of the squarefree power in Remark 2.9 is the squarefree power $(x_1, \ldots, x_n)^2_{sq}$. Separations of this ideal are studied by H.Lohne, [30]. In particular he describes how the separated models are also in bijection with trees on n vertices.

3. Examples of regular quotients of letterplace ideals

The ideals which originally inspired this paper are the multichain ideals of [14].

3.1. **Multichain ideals.** Let P_m be $P \times [m]$ where $m \geq 1$. Consider the surjective map

$$\begin{aligned} [s] \times P_m &\to P \times [m+s-1] \\ (i, p, a) &\mapsto (p, a+i-1). \end{aligned}$$

This map has left strict chain fibers. The image of $L(s, P_m)$ in $\Bbbk[x_{P\times[m+s-1]}]$ is exactly the multichain ideal $I_{m+s-1,s}(P)$ of [14]. This is the ideal generated by monomials

$$x_{p_1,i_1} x_{p_2,i_2} \cdots x_{p_s,i_s}$$

where

$$p_1 \leq \cdots \leq p_s, \quad 1 \leq i_1 < \cdots < i_s \leq m+s-1.$$

By Theorem 2.1 it is obtained from the s'th letterplace ideal $L(s, P_m) = L(s, P \times [m])$ by cutting down by a regular sequence. Thus we recover the fact, [14, Thm. 2.4], that these ideals are Cohen-Macaulay.

The Alexander dual of $L(s, P_m)$ is $L(P_m, s)$. An element r of $\mathrm{Hom}(P \times [m], [s])$ may be represented by sequences

$$1 \leq r_{p1} \leq \cdots \leq r_{pm} \leq s$$

such that for each $p \leq q$ we have $r_{pj} \leq r_{qj}$.

The element r gives the monomial generator in $L(P_m, s)$

$$m_r = \prod_{p \in P} \prod_{i=1}^{m} x_{p,i,r_{pi}}.$$

By Theorem 2.10, the Alexander dual of the multichain ideal $I_{m+s-1,s}(P)$ is then generated by

$$\prod_{p \in P} \prod_{i=1}^{m} x_{p,t_{pi}}, \quad 1 \leq t_{p1} < t_{p2} < \cdots < t_{p_m} \leq m+s-1$$

(where $t_{pi} = r_{pi} + i - 1$) such that $p < q$ implies $t_{pj} \leq t_{qj}$. These are exactly the generators of the squarefree power ideal $L(P, s+m-1)^{\langle m \rangle}$. This recovers Theorem 1.1(b) in [14].

3.2. **Initial ideals of determinantal ideals: two-minors.** We now let $P = [n]$ and $s = 2$. Let $e, f \geq 0$. There are isotone maps

$$[2] \times [n] \times [m] = [2] \times P_m \xrightarrow{\phi_{e,f}} [n+e] \times [m+f]$$
$$(1, a, b) \mapsto (a, b)$$
$$(2, a, b) \mapsto (a+e, b+f)$$

These maps have left strict chain fibers and we get the ideal $L^{\phi_{e,f}}(2, P_m)$.

- When $(e, f) = (0, 1)$ we are in the situation of the previous Subsection 3.1, and we get the multichain ideal $I_{m+1,2}([n])$.
- When $(e, f) = (1, 0)$ we get the multichain ideal $I_{n+1,2}([m])$.
- When $(e, f) = (1, 1)$ we get the ideal in $\Bbbk[x_{[n+1]\times[m+1]}]$ generated by monomials $x_{i,j}x_{i',j'}$ where $i < i'$ and $j < j'$. This is precisely the initial ideal I of the ideal of two-minors of a generic $(n+1) \times (m+1)$ matrix of linear forms $(x_{i,j})$ with respect to a suitable monomial order with respect to a diagonal term order, [36].

In particular all of $I_{m+1,2}([n]), I_{n+1,2}([m])$ and I have the same graded Betti numbers and the same h-vector, the same as $L(2, [n] \times [m])$.

Particularly noteworthy is the following: The ideal of two-minors of the generic $(n+1) \times (m+1)$ matrix is the homogeneous ideal of the Segre product of $\mathbb{P}^m \times \mathbb{P}^n$ in $\mathbb{P}^{nm+n+m}$. By J.Kleppe [28], any deformation of a generic determinantal ideal is still a generic determinantal ideal. So if this Segre embedding is obtained from a variety X in a higher dimensional projective space, by cutting it down by a regular sequence of linear forms, this X must be a cone over the Segre embedding. Thus we cannot "lift" the ideal of two minors to an ideal in a polynomial ring with more variables than $(n+1)(m+1)$. However its initial ideal may be separated to the monomial ideal $L(2, [n] \times [m])$ with $2nm$ variables.

Varying e and f, we get a whole family of ideals $L^{\phi_{e,f}}(2, [n] \times [m])$ with the same Betti numbers as the initial ideal of the ideal of two-minors. When $e = 0 = f$ we get an artinian reduction, not of the initial ideal of the ideal of two-minors, but of its separated model $L(2, [n] \times [m])$. When $e \geq n+1$ and $f \geq m+1$, the map $\phi_{e,f}$ is injective and $L^{\phi_{e,f}}(2, [n] \times [m])$ is isomorphic to the ideal generated by $L(2, [n] \times [m])$ in a polynomial ring with more variables.

3.3. **Initial ideals of determinantal ideals: higher minors.** We may generalize to arbitrary s and two weakly increasing sequences

$$\mathbf{e} = (e_1 = 0, e_2, \ldots, e_s), \quad \mathbf{f} = (f_1 = 0, f_2, \ldots, f_s).$$

We get isotone maps

$$[s] \times [n] \times [m] \longrightarrow [n + e_s] \times [m + f_s]$$
$$(i, a, b) \mapsto (a + e_i, b + f_i)$$

- When $\mathbf{e} = (0, \ldots, 0)$ and $\mathbf{f} = (0, 1, \ldots, s-1)$ we get the multichain ideal $I_{m+s-1,s}([n])$.
- When $\mathbf{e} = (0, 1, \ldots, s-1)$ and $\mathbf{f} = (0, \ldots, 0)$ we get the multichain ideal $I_{n+s-1,s}([m])$.

- When $\mathbf{e} = (0, 1, \ldots, s-1)$ and $\mathbf{f} = (0, 1, \ldots, s-1)$ we get the ideal I generated by monomials
$$x_{i_1,j_1} x_{i_2,j_2} \cdots x_{i_s,j_s}$$
where $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$. This is the initial ideal I of the ideal of s-minors of a general $(n+s-1) \times (m+s-1)$ matrix $(x_{i,j})$ with respect to a diagonal term order, [36].

We thus see that this initial ideal I has a lifting to $L(s, [n] \times [m])$ with snm variables, in contrast to the $(n+s-1)(m+s-1)$ variables which are involved in the ideal of s-minors. We get maximal minors when, say $m = 1$. Then the initial ideal I involves sn variables. So in this case the initial ideal I involves the same number of variables as $L(s, [n])$, i.e. the generators of these two ideals are in one to one correspondence by a bijection of variables.

3.4. **Initial ideal of the ideal of two-minors of a symmetric matrix.** Let $P = \mathrm{Hom}([2], [n])$. The elements here may be identified with pairs (i_1, i_2) where $1 \leq i_1 \leq i_2 \leq n$. There is an isotone map
$$\begin{aligned} \phi : [2] \times \mathrm{Hom}([2], [n]) &\to \mathrm{Hom}([2], [n+1]) \\ (1, i_1, i_1) &\mapsto (i_1, i_2) \\ (2, i_1, i_2) &\mapsto (i_1 + 1, i_2 + 1). \end{aligned}$$
This map has left strict chain fibers, and we get a regular quotient ideal $L^\phi(2, \mathrm{Hom}([2], [n]))$, generated by $x_{i_1,i_2} x_{j_1,j_2}$ where $i_1 < j_1$ and $i_2 < j_2$ (and $i_1 \leq i_2$ and $j_1 \leq j_2$). This is the initial ideal of the ideal generated by 2-minors of a symmetric matrix of size $n+1$, see [5, Sec.5].

3.5. **Ladder determinantal ideals.** Given a poset ideal $\mathcal{J}$ in $[m] \times [n]$. This gives the letterplace ideal $L(2, \mathcal{J})$. There is a map
$$\begin{aligned} \phi : [2] \times \mathcal{J} &\to [m+1] \times [n+1] \\ (1, a, b) &\mapsto (a, b) \\ (2, a, b) &\mapsto (a+1, b+1) \end{aligned}$$
The poset ideal $\mathcal{J}$ is sometimes also called a one-sided ladder in $[m] \times [n]$. The ideal $L^\phi(2, \mathcal{J})$ is the initial ideal of the ladder determinantal ideal associated to $\mathcal{J}$, [34, Cor.3.4]. Hence we recover the fact that these are Cohen-Macaulay, [26, Thm.4.9].

3.6. **Pfaffians.** Let $T(n)$ be the poset in $[n] \times [n]$ consists of all (a, b) with $a + b \leq n + 1$. Then $L^\phi(2, T(n))$ is the initial ideal of the ideal of 4-Pfaffians of a skew-symmetric matrix of rank $n+3$, [26, Sec.5]. It is also the initial ideal of the Grassmannian $G(2, n+3)$, [13, Ch.6].

The poset $T(2)$ is the V poset. The letterplace ideals $L(n, T(2))$ are the initial ideals of the $2n$-Pfaffians of a generic $(2n+1) \times (2n+1)$ skew-symmetric matrix, by [26, Thm.5.1]. The variables $X_{i,2n+2-i}$ in loc.cit. correspond to our variables $x_{i,(1,1)}$ for $i = 1, \ldots, n$, the variables $X_{i+1,2n+2-i}$ correspond to the $x_{i,(2,1)}$ and the $X_{i,2n+1-i}$ to the $x_{i,(1,2)}$.

4. Description of facets and ideals

As we have seen $\mathrm{Hom}(Q, P)$ is itself a poset. The product $P \times Q$ makes the category of posets **Poset** into a symmetric monoidal category, and with this internal Hom, it is a symmetric monoidal closed category [31, VII.7], i.e. there is an adjunction of functors

$$\mathbf{Poset} \underset{\mathrm{Hom}(P,-)}{\overset{-\times P}{\rightleftarrows}} \mathbf{Poset}$$

so that

$$\mathrm{Hom}(Q \times P, R) \cong \mathrm{Hom}(Q, \mathrm{Hom}(P, R)).$$

This is an isomorphism of posets. Note that the distributive lattice $D(P)$ associated to P, consisting of the poset ideals in P, identifies with $\mathrm{Hom}(P, [2])$. In particular $[n+1]$ identifies as $\mathrm{Hom}([n], [2])$. The adjunction above gives isomorphisms between the following posets.

1. $\mathrm{Hom}([m], \mathrm{Hom}(P, [n+1]))$
2. $\mathrm{Hom}([m] \times P, [n+1]) = \mathrm{Hom}([m] \times P, \mathrm{Hom}([n], [2]))$
3. $\mathrm{Hom}([m] \times P \times [n], [2])$
4. $\mathrm{Hom}([n] \times P, \mathrm{Hom}([m], [2])) = \mathrm{Hom}([n] \times P, [m+1])$
5. $\mathrm{Hom}([n], \mathrm{Hom}(P, [m+1]))$

These Hom-posets normally give distinct letterplace or co-letterplace ideals associated to the same underlying (abstract) poset. There are natural bijections between the generators. The degrees of the generators are normally distinct, and so they have different resolutions.

Letting P be the one element poset, we get from 2.,3., and 4. above isomorphisms

$$\mathrm{Hom}([m], [n+1]) \cong \mathrm{Hom}([m] \times [n], [2]) \cong \mathrm{Hom}([n], [m+1]). \tag{1}$$

An element ϕ in $\mathrm{Hom}([m], [n+1])$ identifies as a partition $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ with m parts of sizes $\leq n$, by $\phi(i) = \lambda_{m+1-i} + 1$. The left and right side of the isomorphisms above give the correspondence between a partition and its dual. This poset is the Young lattice. In Stanley's book [35], Chapter 6 is about this lattice, there denoted $L(m, n)$.

Letting $m = 1$ we get by 2.,3., and 5. isomorphims:

$$\mathrm{Hom}(P, [n+1]) \cong \mathrm{Hom}(P \times [n], [2]) \cong \mathrm{Hom}(n, D(P))$$

and so we have ideals

$$L(P, n+1), \quad L(P \times [n], [2]), \quad L(n, D(P))$$

whose generators are naturally in bijection with each other, in particular with elements of $\mathrm{Hom}([n], D(P))$, which are chains of poset ideals in $D(P)$:

$$\emptyset = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n \subseteq \mathcal{J}_{n+1} = P. \tag{2}$$

The facets of the simplicial complexes associated to their Alexander duals

$$L(n+1, P), \quad L(2, P \times [n]), \quad L(D(P), n),$$

are then in bijection with elements of $\mathrm{Hom}([n], D(P))$.

For a subset A of a set R, let A^c denote its complement $R \backslash A$.

1. The facets of the simplicial complex associated to $L(n+1, P)$ identifies as the complements $(\Gamma\phi)^c$ of graphs of $\phi : P \to [n+1]$. This is because these facets correspond to the complements of the set of variables in the generators in the Alexander dual $L(P, n+1)$ of $L(n+1, P)$.

For isotone maps $\alpha : [n+1] \times P \to R$ having bistrict chain fibers, the associated simplicial complex of the ideal $L^\alpha(n+1, P)$, has also facets in one-to-one correspondence with $\phi : P \to [n+1]$, or equivalently $\phi' : [n] \to D(P)$, but the precise description varies according to α.

2. The facets of the simplicial complex associated to $L(2, P \times [n])$ identifies as the complements $(\Gamma\phi)^c$ of the graphs of $\phi : P \times [n] \to [2]$. Alternatively the facets identifies as the graphs $\Gamma\phi'$ of $\phi' : P \times [n] \to [2]^{op}$.

3. Let

$$\alpha : [2] \times P \times [n] \to P \times [n+1], \quad (a, p, i) \mapsto (p, a+i-1).$$

The ideal $L^\alpha(2, P \times [n])$ is the multichain ideal $I_{n+1,2}(P)$. The generators of this ideal are $x_{p,i}x_{q,j}$ where $p \le q$ and $i < j$. The facets of the simplicial complex associated to this ideal are the graphs $\Gamma\phi$ of $\phi : P \to [n+1]^{op}$.

5. Co-letterplace ideals of poset ideals

5.1. **The ideal** $L(P, n; \mathcal{J})$**.** Since $\mathrm{Hom}(P, [n])$ is itself a partially ordered set, we can consider poset ideals $\mathcal{J} \subseteq \mathrm{Hom}(P, [n])$ and form the subideal $L(P, n; \mathcal{J})$ of $L(P, n)$ generated by the monomials $m_{\Gamma\phi}$ where $\phi \in \mathcal{J}$. We call it the *co-letterplace ideal of the poset ideal* $\mathcal{J}$. For short we often write $L(\mathcal{J})$ and call it simply a co-letterplace ideal. For the notion of linear quotients we refer to [22].

Theorem 5.1. *Let* $\mathcal{J}$ *be a poset ideal in* $\mathrm{Hom}(P, [n])$*. Then* $L(P, n; \mathcal{J})$ *has linear quotients, and so it has linear resolution.*

Proof. We extend the partial order $\le$ on $\mathcal{J}$ to a total order, denoted $\le^t$, and define an order on the generators of $L(\mathcal{J})$ be setting $m_{\Gamma\psi} \ge m_{\Gamma\phi}$ if and only if $\psi \le^t \phi$. We claim that $L(\mathcal{J})$ has linear quotients with respect to this total order of the monomial generators of $L(\mathcal{J})$. Indeed, let $m_{\Gamma\psi} > m_{\Gamma\phi}$ where $\psi \in \mathcal{J}$. Then $\psi <^t \phi$, and hence there exists $p \in P$ such that $\psi(p) < \phi(p)$. We choose a $p \in P$ which is minimal with this property. Therefore, if $q < p$, then $\phi(q) \le \psi(q) \le \psi(p) < \phi(p)$. We set

$$\psi'(r) = \begin{cases} \psi(r), & \text{if } r = p, \\ \phi(r), & \text{otherwise.} \end{cases}$$

Then $\psi' \in \mathrm{Hom}(P, n)$ and $\psi' < \phi$ for the original order on P. It follows that $\psi' \in \mathcal{J}$, and $m_{\Gamma\psi'} > m_{\Gamma\phi}$. Since $(m_{\Gamma\psi'}) : m_{\Gamma\phi} = (x_{p,\psi(p)})$ and since $x_{p,\psi(p)}$ divides $m_{\Gamma\psi}$, the desired conclusion follows. ☐

Remark 5.2. One may fix a maximal element $p \in P$. The statement above still holds if $\mathcal{J}$ in $\mathrm{Hom}(P, [n])$ is a poset ideal for the weaker partial order $\le^w$ on $\mathrm{Hom}(P, [n])$ where $\phi \le^w \psi$ if $\phi(q) \le \psi(q)$ for $q \ne p$ and $\phi(p) = \psi(p)$. Just let the total order still refine the standard partial order on the $\mathrm{Hom}(P, [n])$. Then one deduces either $\psi' \le^w \phi$ or $\psi' \le^w \psi$. In either case this gives $\psi' \in \mathcal{J}$.

For an isotone map $\phi : P \to [n]$, we define the set

$$\Lambda\phi = \{(p, i) \mid \phi(q) \le i < \phi(p) \text{ for all } q < p\}. \quad (3)$$

It will in the next subsection play a role somewhat analogous to the graph $\Gamma\phi$. For $\phi \in \mathcal{J}$ we let J_ϕ be the ideal generated by all $m_{\Gamma\psi}$ with $m_{\Gamma\psi} > m_{\Gamma\phi}$, where we use the total order in the proof of Theorem 5.1 above. In analogy to [14, Lemma 3.1] one obtains:

Corollary 5.3. *Let $\phi \in \mathcal{J}$. Then $J_\phi : m_{\Gamma\phi}$ is $\{x_{p,i} \,|\, (p,i) \in \Lambda\phi\}$.*

Proof. The inclusion $\subseteq$ has been shown in the proof of Theorem 5.1. Conversely, let $x_{p,i}$ be an element of the right hand set. We set

$$\psi(r) = \begin{cases} i, & \text{if } r = p, \\ \phi(r), & \text{otherwise.} \end{cases}$$

Then $m_{\Gamma\psi} \in J_\phi$ and $(m_{\Gamma\psi}) : m_{\Gamma\phi} = (x_{p,i})$. This proves the other inclusion. □

Corollary 5.4. *The projective dimension of $L(P, n; \mathcal{J})$ is the maximum of the cardinalities $|\Lambda\phi|$ for $\phi \in \mathcal{J}$.*

Proof. This follows by the above Corollary 5.3 and Lemma 1.5 of [25]. □

Remark 5.5. By [14, Cor.3.3] the projective dimension of $L(P, n)$ is $(n-1)s$ where s is the size of a maximal antichain in P. It is not difficult to work this out as a consequence of the above when $\mathcal{J} = \mathrm{Hom}(P, [n])$.

An explicit form of the minimal free resolution of $L(P, n)$ is given in [14, Thm. 3.6], and this is generalized to $L(P, n; \mathcal{J})$ in [8].

5.2. **Regular quotients of $L(P, n; \mathcal{J})$.** We now consider co-letterplace ideals of poset ideals when we cut down by a regular sequence of variable differences. The following generalizes Theorem 2.2 and we prove it in Section 8.

Theorem 5.6. *Given an isotone map $\psi : P \times [n] \to R$ with left strict chain fibers. Let $\mathcal{J}$ be a poset ideal in $\mathrm{Hom}(P, n)$. Then the basis B (as defined before Theorem 2.1) is a regular sequence for the ring $\Bbbk[x_{P\times[n]}]/L(P, n; \mathcal{J})$.*

5.3. **Alexander dual of $L(P, n; \mathcal{J})$.** We describe the Alexander dual of $L(\mathcal{J}) = L(P, n; \mathcal{J})$ when $\mathcal{J}$ is a poset ideal in $\mathrm{Hom}(P, [n])$. We denote this Alexander dual ideal as $L(\mathcal{J})^A = L(n, P; \mathcal{J})$. Note that since $L(P, n; \mathcal{J}) \subseteq L(P, n)$, the Alexander dual $L(n, P; \mathcal{J})$ contains the letterplace ideal $L(n, P)$, and since $L(P, n; \mathcal{J})$ has linear resolution, the Alexander dual $L(n, P; \mathcal{J})$ is a Cohen-Macaulay ideal, by [12]. Recall the set $\Lambda\phi$ defined above (3), associated to a map $\phi \in \mathrm{Hom}(P, [n])$.

Lemma 5.7. *Let $\mathcal{J}$ be a poset ideal in $\mathrm{Hom}(P, [n])$. Let $\phi \in \mathcal{J}$ and ψ be in the complement $\mathcal{J}^c$. Then $\Lambda\psi \cap \Gamma\phi$ is nonempty.*

Proof. There is some $p \in P$ with $\psi(p) > \phi(p)$. Choose p to be minimal with this property, and let $i = \phi(p)$. If (p, i) is not in $\Lambda\psi$, there must be $q < p$ with $\psi(q) > i = \phi(p) \geq \phi(q)$. But this contradicts p being minimal. Hence $(p, i) = (p, \phi(p))$ is both in $\Gamma\phi$ and $\Lambda\psi$. □

Lemma 5.8. *Let S be a subset of $P \times [n]$ which is disjoint from $\Gamma\phi$ for some ϕ in $\mathrm{Hom}(P, [n])$. If ϕ is a minimal such element w.r.t. the partial order on $\mathrm{Hom}(P, [n])$, then $S \supseteq \Lambda\phi$.*

Proof. Suppose $(p, i) \in \Lambda\phi$ and (p, i) is not in S. Define $\phi' : P \to [n]$ by

$$\phi'(q) = \begin{cases} \phi(q), & q \neq p \\ i, & q = p \end{cases}$$

By definition of $\Lambda\phi$ we see that ϕ' is an isotone map, and $\phi' < \phi$. But since S is disjoint from $\Gamma\phi$, we see that it is also disjoint from $\Gamma\phi'$. This contradicts ϕ being minimal. Hence every $(p, i) \in \Lambda\phi$ is also in S. $\square$

For a subset $\mathcal{S}$ of $\mathrm{Hom}(P, [n])$ define $K(\mathcal{S}) \subseteq \Bbbk[x_{[n]\times P}]$ to be the ideal generated by the monomials $m_{\Lambda\phi^\tau}$ where $\phi \in \mathcal{S}$.

Theorem 5.9. *The Alexander dual* $L(n, P; \mathcal{J})$ *is* $L(n, P) + K(\mathcal{J}^c)$*. This is a Cohen-Macaulay ideal of codimension* $|P|$*.*

Proof. It is Cohen-Macaulay, in fact shellable, since $L(\mathcal{J})$ has linear quotients by Theorem 5.1. The facets of the simplicial complex corresponding to $L(\mathcal{J})^A$ are the complements of the generators of $L(\mathcal{J})$. Hence these facets have codimension $|P|$.

To prove the first statement we show the following.

1. The right ideal is contained in the Alexander dual of the left ideal: Every monomial in $L(n, P) + K(\mathcal{J}^c)$ has non-trivial common divisor with every monomial in $L(\mathcal{J})$.

2. The Alexander dual of the left ideal is contained in the right ideal: If $S \subseteq [n] \times P$ intersects every $\Gamma\phi^\tau$ where $\phi \in \mathcal{J}$, the monomial m_S is in $L(n, P) + K(\mathcal{J}^c)$.

1a. Let $\psi \in \mathrm{Hom}([n], P)$. Since $L(n, P)$ and $L(P, n)$ are Alexander dual, $\Gamma\psi \cap \Gamma\phi^\tau$ is non-empty for every $\phi \in \mathrm{Hom}(P, [n])$ and so in particular for every $\phi \in \mathcal{J}$.
1b. If $\psi \in \mathcal{J}^c$ then $\Lambda\psi \cap \Gamma\phi$ is nonempty for every $\phi \in \mathcal{J}$ by Lemma 5.7.

Suppose now S intersects every $\Gamma\phi^\tau$ where ϕ is in $\mathcal{J}$.
2a. If S intersects every $\Gamma\phi^\tau$ where ϕ is in $\mathrm{Hom}(P, [n])$, then since $L(n, P)$ is the Alexander dual of $L(P, n)$, the monomial m_S is in $L(n, P)$.
2b. If S does not intersect $\Gamma\phi^\tau$ where $\phi \in \mathcal{J}^c$, then by Lemma 5.8, for a minimal such ϕ we will have $S \supseteq \Lambda\phi^\tau$. Since S intersects $\Gamma\phi^\tau$ for all $\phi \in \mathcal{J}$, a minimal such ϕ is in $\mathcal{J}^c$. Thus m_S is divided by $m_{\Lambda\phi^\tau}$ in $K(\mathcal{J}^c)$. $\square$

Remark 5.10. For a more concrete example, to Stanley-Reisner ideals with whiskers, see the end of Subsection 6.4.

Remark 5.11. In [8, Thm.5.1] it is shown that the simplicial complex corresponding to $L(n, P; \mathcal{J})$ is a triangulation of a ball. Its boundary is then a triangulation of a sphere. This gives a comprehensive generalization of Bier spheres, [3]. In [8, Sec.4] there is also a precise description of the canonical module of the Stanley-Reisner ring of $L(n, P; \mathcal{J})$, as an ideal in this Stanley-Reisner ring.

5.4. **Regular quotients of the Alexander dual** $L(n, P; \mathcal{J})$**.** We now take the Alexander dual of $L(P, n; \mathcal{J})$ and cut it down by a regular sequence of variable differences. We then get a generalization of Theorem 2.1 and we prove it in Section 8.

Theorem 5.12. *Given an isotone map* $\phi : [n] \times P \to R$ *with left strict chain fibers. Let* $\mathcal{J}$ *be a poset ideal in* $\mathrm{Hom}(P, n)$*. Then the basis* B *(as defined before Theorem 2.1) is a regular sequence for the ring* $\Bbbk[x_{[n]\times P}]/L(n, P; \mathcal{J})$*.*

6. Examples of regular quotients of co-letterplace ideals

We give several examples of quotients of co-letterplace ideals which have been studied in the literature in recent years.

6.1. **Strongly stable ideals:** *Poset ideals in* $\mathrm{Hom}([d],[n])$.

Elements of $\mathrm{Hom}([d],[n])$ are in one to one correspondence with monomials in $\Bbbk[x_1,\ldots,x_n]$ of degree d: A map ϕ gives the monomial $\Pi_{i=1}^d x_{\phi(i)}$. By this association, the poset ideals in $\mathrm{Hom}([d],[n])$ are in one to one correspondence with strongly stable ideals in $\Bbbk[x_1,\ldots,x_n]$ generated in degree d.

Consider the projections $[d]\times[n] \xrightarrow{p_2} [n]$. The following is a consequence of Theorem 5.1 and Theorem 5.6.

Corollary 6.1. *Let $\mathcal{J}$ be a poset ideal of $\mathrm{Hom}([d],[n])$. Then $L(\mathcal{J})$ has linear resolution. The quotient map*

$$\Bbbk[x_{[d]\times[n]}]/L(\mathcal{J}) \xrightarrow{\hat{p_2}} \Bbbk[x_{[n]}]/L^{p_2}(\mathcal{J})$$

is a quotient map by a regular sequence, and $L^{p_2}(\mathcal{J})$ is the strongly stable ideal in $\Bbbk[x_1,\ldots,x_n]$ associated to $\mathcal{J}$.

The ideals $L(\mathcal{J})$ are extensively studied by Nagel and Reiner in [33]. Poset ideals $\mathcal{J}$ of $\mathrm{Hom}([d],[n])$ are there called strongly stable d-uniform hypergraphs, [33, Def. 3.3]. If M is the hypergraph corresponding to $\mathcal{J}$, the ideal $L(\mathcal{J})$ is the ideal $I(F(M))$ of the d-partite d-uniform hypergraph $F(M)$ of [33, Def.3.4, Ex.3.5].

Furthermore the ideal $L^{p_2}(\mathcal{J})$ is the ideal $I(M)$ of [33, Ex. 3.5]. The squarefree ideal $I(K)$ of [33, Ex.3.5] is the ideal $L^{\phi}(\mathcal{J})$ obtained from the map:

$$\begin{aligned} \phi : [d]\times[n] &\to [d+n-1] \\ (a,b) &\mapsto a+b-1 \end{aligned}$$

Corollary 6.1 above is a part of [33, Thm. 3.13].

Given a sequence $0 = a_0 \leq a_1 \leq \cdots \leq a_{d-1}$, we get an isotone map

$$\begin{aligned} \alpha : [d]\times[n] &\to [n+a_{d-1}] \\ (i,j) &\mapsto j + a_{i-1} \end{aligned}$$

having left strict chain fibers. The ideal $L^{\alpha}(\mathcal{J})$ is the ideal coming from the strongly stable ideal associated to $\mathcal{J}$ by the stable operator of S.Murai [32, p.707]. When $a_{i-1} < a_i$ they are called alternative squarefree operators in [37, Sec. 4].

Remark 6.2. In [18] Francisco, Mermin and Schweig consider a poset Q with underlying set $\{1,2,\ldots,n\}$ where Q is a weakening of the natural total order, and study Q-Borel ideals. This is not quite within our setting, but adds extra structure: Isotone maps $\phi : [d] \to [n]$ uses the total order on $[n]$ but when studying poset ideals $\mathcal{J}$ the weaker poset structure Q is used on the codomain.

Let $\underline{n}$ be the poset which is the disjoint union of the one element posets $\{1\},\ldots\{n\}$, so any two distinct elements are incomparable. This is the antichain on n elements.

6.2. **Ferrers ideals:** *Poset ideals in* $\mathrm{Hom}(\underline{2},[n])$. By (1) partitions $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ where $\lambda_1 \leq n$ correspond to elements of:

$$\mathrm{Hom}([n],[n+1]) \cong \mathrm{Hom}([n]\times[n],[2]).$$

Thus λ gives a poset ideal $\mathcal{J}$ in $[n]\times[n] = \mathrm{Hom}(\underline{2},[n])$. The Ferrers ideal I_λ of [7, Sec. 2] is the ideal $L(\mathcal{J})$ in $\Bbbk[x_{\underline{2}\times[n]}]$. In particular we recover the result from [7, Cor.3.8] that it has linear resolution.

More generally, the poset ideals $\mathcal{J}$ of $\mathrm{Hom}(\underline{d}, [n])$ correspond to the d-partite d-uniform Ferrers hypergraphs F in [33, Def. 3.6]. That $L(\mathcal{J})$ has linear resolution is [33, Thm. 3.13].

6.3. **Edge ideals of cointerval d-hypergraphs.** Let $\mathrm{Hom}_s(Q, P)$ be strict isotone maps ϕ, i.e. $q < q'$ implies $\phi(q) < \phi(q')$. There is an isomorphism of posets

$$\mathrm{Hom}([d], [n]) \cong \mathrm{Hom}_s([d], [n + d - 1]), \tag{4}$$

by sending ϕ to ϕ_s given by $\phi_s(j) = \phi(j) + j - 1$.

Consider the weaker partial order on $\preceq$ on $\mathrm{Hom}([d], [n])$ where $\phi \preceq \psi$ if $\phi(i) \leq \psi(i)$ for $i < d$ and $\phi(d) = \psi(d)$. Via the isomorphism (4) this gives a partial order $\preceq_s$ on $\mathrm{Hom}_s([d], [n + d - 1])$. The poset ideals for the partial order $\preceq_s$ correspond to the cointerval d-hypergraphs of [10, Def. 4.1] on the set $\{1, 2, \ldots, n + d - 1\}$. Let $\mathcal{J}_s$ be such a poset ideal for $\preceq_s$. It corresponds to a poset ideal $\mathcal{J}$ in $\mathrm{Hom}([d], [n])$ for $\preceq$. Let

$$\begin{aligned} \phi : [d] \times [n] &\to [d + n - 1] \\ (a, b) &\mapsto a + b - 1 \end{aligned} \tag{5}$$

The ideal $L^\phi(\mathcal{J})$ is the edge ideal of the cointerval hypergraph corresponding to $\mathcal{J}_s$, see [10, Def. 2.1]. By remarks 5.2 and 8.2, theorems 5.6 and 5.1 still holds for the weaker partial order $\preceq$. Hence we recover the fact from [10, Cor. 4.7] that edge ideals of cointerval hypergraphs have linear resolution. In the case $d = 2$ these ideals are studied also in [7, Sec. 4] and [33, Sec. 2]. These are obtained by cutting down by a regular sequence of differences of variables from a *skew* Ferrers ideals $I_{\lambda-\mu}$. The skewness implies the ideal comes from a poset ideal of $\mathrm{Hom}([2], [n])$ rather than $\mathrm{Hom}(\underline{2}, [n])$. Due to this we get the map (5) which has left strict chain fibers, and so the ideal $\overline{I_{\lambda-\mu}}$, of [7, Sec. 4].

6.4. **Uniform face ideals:** *Poset ideals in* $\mathrm{Hom}(\underline{n}, [2])$. The uniform face ideal of a simplicial complex Δ, introduced recently by D.Cook [6], see also [20], is the ideal generated by the monomials

$$\prod_{i \in F} x_i \cdot \prod_{i \notin F} y_i$$

as F varies among the faces of Δ. The Boolean poset on n elements is the distributive lattice $D(\underline{n}) = \mathrm{Hom}(\underline{n}, [2])$. A simplicial complex Δ on the set $\{1, 2, \ldots, n\}$ corresponds to a poset ideal $\mathcal{J}$ of $\mathrm{Hom}(\underline{n}, [2])$, and the uniform face ideal of Δ identifies as the subideal $L(\mathcal{J})$ of $L(\underline{n}, [2])$.

More generally Cook considers a set of vertices which is a disjoint union of k ordered sets $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k$, each $\mathcal{C}_i$ considered a colour class. He then considers simplicial complexes Δ which are *nested* with respect to these orders [6, Def.3.3, Prop.3.4]. He associates to this the *uniform face ideal* $I(\Delta, \mathcal{C})$, [6, Def. 4.2]. Let c_i be the cardinality of $\mathcal{C}_i$ and consider the poset which is the disjoint union $\cup_{i=1}^k [c_i]$. Then such a Δ corresponds precisely to a poset ideal $\mathcal{J}$ in $\mathrm{Hom}(\cup_1^k [c_i], [2])$. In fact $\mathcal{J}$ is isomorphic to the index poset $P(\Delta, \mathcal{C})$ of [6, Def. 6.1]. The uniform face ideal is obtained as follows: There are projection maps $p_i : [c_i] \times [2] \to [2]$ and so

$$\cup_1^k p_i : (\cup_1^k [c_i]) \times [2] \to \cup_1^k [2].$$

This map has left strict chain fibers and the ideal $L^{\cup_1^k p_i}(\cup_1^k [c_i], [2])$ is exactly the uniform face ideal $I(\Delta, \mathcal{C})$. In [6, Thm. 6.8] it is stated that this ideal has linear resolution.

Returning again to the first case of the ideal $L(\mathcal{J})$ in $L(\underline{n}, [2])$, its Alexander dual is by Theorem 5.9:

$$L(\mathcal{J})^A = L([2], \underline{n}) + K(\mathcal{J}^c).$$

Here $L([2], \underline{n})$ is the complete intersection of $x_{1j}x_{2j}$ for $j = 1, \ldots, n$, while $K(\mathcal{J}^c)$ is generated by $\prod_{j \in G} x_{1j}$ where G is a nonface of Δ. Thus $K(\mathcal{J}^c)$ is the associated ideal $I_\Delta \subseteq \Bbbk[x_{11}, \ldots, x_{1n}]$. This is [20, Thm.1.1]: $L(\mathcal{J})^A$ is the Stanley-Reisner ideal I_Δ with whiskers $x_{1j}x_{2j}$.

7. Proof concerning Alexander duality

In this section we prove Theorem 2.10 concerning the compatibility between Alexander duality and cutting down by a regular sequence. The following lemma holds for squarefree ideals. Surprisingly it does not hold for monomial ideals in general, for instance for $(x_0^n, x_1^n) \subseteq k[x_0, x_1]$.

Lemma 7.1. *Let $I \subseteq S$ be a squarefree monomial ideal and let $f \in S$ be such that $x_1 f = x_0 f$ considered in S/I. Then for every monomial m in f we have $x_1 m = 0 = x_0 m$ in S/I.*

Proof. Write $f = x_0^a f_a + \cdots x_0 f_1 + f_0$ where each f_i does not contain x_0. The terms in $(x_1 - x_0)f = 0$ of degree $(a+1)$ in x_0, are in $x_0^{a+1} f_a$, and so this is zero. Since S/I is squarefree, $x_0 f_a$ is zero, and so $f = x_0^{a-1} f_{a-1} + \cdots$. We may continue and get $f = f_0$. But then again in $(x_1 - x_0)f = 0$ the terms with x_0 degree 1 is $x_0 f_0$ and so this is zero. The upshot is that $x_0 f = 0 = x_1 f$. But then each of the multigraded terms of these must be zero, and this gives the conclusion. $\square$

Let S be the polynomial ring $\Bbbk[x_0, x_1, x_2, \ldots, x_n]$ and $I \subseteq S$ a squarefree monomial ideal with Alexander dual $J \subseteq S$. Let $S_1 = k[x, x_2, \ldots, x_n]$ and $S \to S_1$ be the map given by $x_i \mapsto x_i$ for $i \geq 2$ and $x_0, x_1 \mapsto x$.

Let I_1 be the ideal of S_1 which is the image of I, so the quotient ring of S/I by the element $x_1 - x_0$ is the ring S_1/I_1. Similarly we define J_1. We now have the following. Part c. below is Theorem 3.1 in the unpublished paper [30].

Proposition 7.2. *a) If $x_1 - x_0$ is a regular element of S/I, then J_1 is squarefree.*

b) If I_1 is squarefree then $x_1 - x_0$ is a regular element on S/J.

c) If both $x_1 - x_0$ is a regular element on S/I and I_1 is squarefree, then J_1 is the Alexander dual of I_1.

Proof. The Alexander dual J of I consists of all monomials in S with non-trivial common factor (ntcf.) with all monomials in I.

a) Let F be a facet of the simplicial complex of I. Let $m_F = \prod_{i \in F} x_i$. Suppose F does not contain any of the vertices 0 and 1. Then $x_1 m_F = 0 = x_0 m_F$ in S/I (since F is a facet). Since $x_1 - x_0$ is regular we get $m_F = 0$ in S/I, a contradiction. Thus every facet F contains either 0 or 1. The generators of J are $\prod_{i \in [n] \setminus F} x_i$, and so no such monomial contains $x_0 x_1$ and therefore J_1 will be squarefree.

b) Suppose $(x_1 - x_0)f = 0$ in S/J. By the above for the monomials m in f, we have $x_1 m = 0 = x_0 m$ in S/J. We may assume m is squarefree. So $x_0 m$ has ntcf. with all monomials in I and the same goes for $x_1 m$. If m does not have ntcf.

with the minimal monomial generator n in I, we must then have $n = x_0x_1n'$. But then the image of n in I_1 would not be squarefree, contrary to the assumption. The upshot is that m has ntcf. with every monomial in I and so is zero in S/J.

c) A monomial m in J has ntcf. with all monomials in I. Then its image $\overline{m}$ in S_1 has ntcf. with all monomials in I_1, and so J_1 is contained in the Alexander dual of I_1.

Assume now $\overline{m}$ in S_1 has ntcf. with all monomials in I_1. We must show that $\overline{m} \in J_1$. If $\overline{m}$ does not contain x then m has ntcf. with every monomial in I, and so $\overline{m} \in J_1$.

Otherwise $\overline{m} = x\overline{m'}$ and so $\overline{m} \in J_1$. We will show that either x_0m' or x_1m' is in J. If not, then x_0m' has no common factor with some monomial x_1n_1 in I (it must contain x_1 since $\overline{m}$ has ntcf. with every monomial in I_1), and x_1m' has no common factor with some monomial x_0n_0 in I. Let n be the least common multiple of n_0 and n_1. Then x_0n and x_1n are both in I and so by the regularity assumption $n \in I$. But n has no common factor with x_0m' and x_1m', and so $\overline{n} \in I_1$ has no common factor with $\overline{m} = x\overline{m'}$. This is a contradiction. Hence either x_0m' or x_1m' is in J and so $\overline{m}$ is in J_1. ☐

We are ready to round off this section by the following extension of Theorem 2.10.

Theorem 7.3. *Let $\phi : [n] \times P \to R$ be an isotone map such that the fibers $\phi^{-1}(r)$ are bistrict chains in $[n] \times P^{op}$. Then the ideals $L^\phi(n, P; \mathcal{J})$ and $L^{\phi^\tau}(P, n; \mathcal{J})$ are Alexander dual.*

Proof. Since $L(P, n)$ is squarefree, the subideal $L(P, n; \mathcal{J})$ is also. Furthermore it is obtained by cutting down by a regular sequence of variable differences, by Theorem 5.12. Hence $L(n, P; \mathcal{J})^\phi$ is the Alexander dual of $L(P, n; \mathcal{J})^{\phi^\tau}$. ☐

8. Proof that the poset maps induce regular sequences.

To prove Theorems 2.1, 2.2 and 5.6, we will use an induction argument. Let $[n] \times P \xrightarrow{\phi} R$ be an isotone map. Let $r \in R$ have inverse image by ϕ of cardinality ≥ 2. Choose a partition into nonempty subsets $\phi^{-1}(r) = R_1 \cup R_2$ such that $(i, p) \in R_1$ and $(j, q) \in R_2$ implies $i < j$. Let R' be $R \backslash \{r\} \cup \{r_1, r_2\}$. We get the map

$$[n] \times P \xrightarrow{\phi'} R' \to R \tag{6}$$

factoring ϕ, where the elements of R_i map to r_i. For an element p' of R', denote by $\overline{p'}$ its image in R. Let p', q' be distinct elements of R'. We define a partial order on R' by the following two types of strict inequalities:

a. $p' < q'$ if $p' = r_1$ and $q' = r_2$,
b. $p' < q'$ if $\overline{p'} < \overline{q'}$

Lemma 8.1. *This ordering is a partial order on R'.*

Proof. Transitivity: Suppose $p' \leq q'$ and $q' \leq r'$. Then $\overline{p'} \leq \overline{q'}$ and $\overline{q'} \leq \overline{r'}$ and so $\overline{p'} \leq \overline{r'}$. If either $\overline{p'}$ or $\overline{r'}$ is distinct from r we conclude that $p' \leq r'$. If both of them are equal to r, then $\overline{q'} = r$ also. Then either $p' = q' = r'$ or $p' = r_1$ and $r' = r_2$, and so $p' \leq r'$.

Reflexivity: Suppose $p' \leq q'$ and $q' \leq p'$. Then $\overline{p'} = \overline{q'}$. If this is not r we get $p' = q'$. If it equals r, then since we do not have $r_2 \leq r_1$, we must have again have $p' = q'$. ☐

Proof of Theorem 2.1. We show this by induction on the cardinality of $\operatorname{im}\phi$. Assume that we have a factorization (6), such that

$$\Bbbk[x_{R'}]/L^{\phi'}(n,P) \tag{7}$$

is obtained by cutting down from $\Bbbk[x_{[n]\times P}]/L(n,P)$ by a regular sequence of variable differences.

For (a,p) in $[n]\times P$ denote its image in R' by $\overline{a,p}$ and its image in R by $\overline{\overline{a,p}}$. Let (a,p) map to $r_1 \in R'$ and (b,q) map to $r_2 \in R'$. We will show that $x_{r_1} - x_{r_2}$ is a regular element in the quotient ring (7). So let f be a polynomial of this quotient ring such that $f(x_{r_1} - x_{r_2}) = 0$. Then by Lemma 7.1, for any monomial m in f we have $mx_{r_1} = 0 = mx_{r_2}$ in the quotient ring (7). We may assume m is nonzero in this quotient ring.

There is a monomial $x_{\overline{1,p_1}}x_{\overline{1,p_2}}\cdots x_{\overline{n,p_n}}$ in $L^{\phi'}(n,P)$ dividing $mx_{\overline{a,p}}$ considered as monomials in $\Bbbk[x_R]$. Then we must have $\overline{a,p} = \overline{s,p_s}$ for some s. Furthermore there is $x_{\overline{1,q_1}}x_{\overline{1,q_2}}\cdots x_{\overline{n,q_n}}$ in $L^{\phi'}(n,P)$ dividing $mx_{\overline{b,q}}$ in $\Bbbk[x_R]$, and so $\overline{b,q} = \overline{t,q_t}$ for some t.

In R we now get

$$\overline{\overline{s,p_s}} = \overline{\overline{a,p}} = \overline{\overline{b,q}} = \overline{\overline{t,q_t}},$$

so $s = t$ would imply $q_t = p_s$ since ϕ has left strict chain fibers. But then

$$r_1 = \overline{a,p} = \overline{s,p_s} = \overline{t,q_t} = \overline{b,q} = r_2$$

which is not so. Assume, say $s < t$. Then $p_s \geq q_t$ since ϕ has left strict chain fibers, and so

$$p_t \geq p_s \geq q_t \geq q_s.$$

1. Suppose $p_s > q_t$. Consider $x_{\overline{1,q}}\cdots x_{\overline{t-1,q_{t-1}}}$. This will divide m since $x_{\overline{1,q_1}}x_{\overline{1,q_2}}\cdots x_{\overline{n,q_n}}$ divides $mx_{\overline{t,q_t}}$. Similarly $x_{\overline{s+1,p_{s+1}}}\cdots x_{\overline{n,p_n}}$ divides m. Chose $s \leq r \leq t$. Then $p_r \geq p_s > q_t \geq q_r$ and so $\overline{r,q_r} < \overline{r,p_r}$. Then $x_{\overline{1,q}}\cdots x_{\overline{r-1,q_{r-1}}}$ and $x_{\overline{r,p_r}}\cdots x_{\overline{n,p_n}}$ do not have a common factor since

$$\overline{i,q_i} \leq \overline{r,q_r} < \overline{r,p_r} \leq \overline{j,p_j}$$

for $i \leq r \leq j$. Hence the product of these monomials will divide m and so $m = 0$ in the quotient ring $\Bbbk[x_R]/L^{\phi}(n,P)$.

2. Assume $p_s = q_t$ and $q_t > q_s$. Then $\overline{s,p_s} > \overline{s,q_s}$ since ϕ has left strict chain fibers. The monomials $x_{\overline{1,q_1}}\cdots x_{\overline{s,q_s}}$ and $x_{\overline{s+1,p_{s+1}}}\cdots x_{\overline{n,p_n}}$ then do not have any common factor, and the product divides m, showing that $m = 0$ in the quotient ring $\Bbbk[x_R]/L^{\phi}(n,P)$.

If $p_t > p_s$ we may argue similarly.

3. Assume now that $p_t = p_s = q_t = q_s$, and denote this element as p. Note that for $s \leq i \leq t$ we then have $p_s \leq p_i \leq p_t$, so $p_i = p$, and the same argument shows that $q_i = p$ for i in this range.

Since

$$\overline{s,p} = \overline{s,p_s} = \overline{a,p} \neq \overline{b,q} = \overline{t,q_t} = \overline{t,p}$$

there is $s \leq r \leq t$ such that

$$\overline{s,p_s} = \cdots = \overline{r,p_r} < \overline{r+1,p_{r+1}} \leq \cdots \leq \overline{t,p_t}.$$

This is the same sequence as

$$\overline{s,q_s} = \cdots = \overline{r,q_r} < \overline{r+1,q_{r+1}} \leq \cdots \leq \overline{t,q_t}.$$

Then $x_{\overline{1,q_1}} \cdots x_{\overline{r,q_r}}$ and $x_{\overline{r+1,p_{r+1}}} \cdots x_{\overline{n,p_n}}$ divide m and do not have a common factor, and so $m = 0$ in the quotient ring $\Bbbk[x_R]/L^\phi(n,P)$. □

Proof of Theorem 5.12. There is a surjection

$$\Bbbk[x_{[n]\times P}]/L(n,P) \to \Bbbk[x_{[n]\times P}]/L(\mathcal{J})^A \tag{8}$$

and both are Cohen-Macaulay quotient rings of $\Bbbk[x_{[n]\times P}]$ of codimension $|P|$. The basis B is a regular sequence for the first ring by the previous argument. Hence

$$\Bbbk[x_{[n]\times P}]/(L(n,P) + (B))$$

has codimension $|P| + |B|$. Since

$$\Bbbk[x_{[n]\times P}]/(L(\mathcal{J})^A + (B))$$

is a quotient of this it must have codimension $\geq |P| + |B|$. But then B must be a regular sequence for the right side of (8) also. □

Proof of Theorems 2.2 and 5.6. By induction on the cardinality of $\operatorname{im}\phi$. We assume we have a factorization

$$P \times [n] \xrightarrow{\phi'} R' \to R$$

analogous to (6), such that

$$\Bbbk[x_{R'}]/L^{\phi'}(\mathcal{J}) \tag{9}$$

is obtained by cutting down from $\Bbbk[x_{P\times[n]}]/L(\mathcal{J})$ by a regular sequence of variable differences.

Let (p_0, a) map to $r_1 \in R'$ and (q_0, b) map to $r_2 \in R'$. We will show that $x_{r_1} - x_{r_2}$ is a regular element in the quotient ring (9). So let f be a polynomial of this quotient ring such that $f(x_{r_1} - x_{r_2}) = 0$. Then by Lemma 7.1, for any monomial m in f we have $mx_{r_1} = 0 = mx_{r_2}$ in the quotient ring $\Bbbk[x_{R'}]/L^{\phi'}(\mathcal{J})$. We may assume m is nonzero in this quotient ring.

There is $i \in \mathcal{J} \subseteq \operatorname{Hom}(P,n)$ such that the monomial $m^i = \prod_{p\in P} x_{\overline{p,i_p}}$ in $L^{\phi'}(\mathcal{J})$ divides $mx_{\overline{p_0,a}}$, and similarly a $j \in \mathcal{J}$ such that the monomial $m^j = \prod_{p\in P} x_{\overline{p,j_p}}$ divides $mx_{\overline{q_0,b}}$. Hence there are s and t in P such that $\overline{s,i_s} = \overline{p_0,a}$ and $\overline{t,j_t} = \overline{q_0,b}$. In R we then get:

$$\overline{\overline{s,i_s}} = \overline{\overline{p_0,a}} = \overline{\overline{q_0,b}} = \overline{\overline{t,j_t}},$$

so $s = t$ would imply $i_t = j_t$ since ϕ has left strict chain fibers. But then

$$r_1 = \overline{p_0,a} = \overline{s,i_s} = \overline{t,j_t} = \overline{q_0,b} = r_2$$

which is not so. Assume then, say $s < t$. Then $i_s \geq j_t$ since ϕ has left strict chain fibers, and so

$$i_t \geq i_s \geq j_t \geq j_s. \tag{10}$$

Now form the monomials

- $m^i_{>s} = \prod_{p>s} x_{\overline{p,i_p}}$.
- $m^i_{i>j} = \displaystyle\prod_{i_p>j_p, not\,(p>s)} x_{\overline{p,i_p}}$.
- $m^i_{i<j} = \displaystyle\prod_{i_p<j_p, not\,(p>s)} x_{\overline{p,i_p}}$.
- $m^i_{i=j} = \displaystyle\prod_{i_p=j_p, not\,(p>s)} x_{\overline{p,i_p}}$.

Similarly we define m^j_* for the various subscripts $*$. Then

$$m^i = m^i_{i=j} \cdot m^i_{i>j} \cdot m^i_{i<j} \cdot m^i_{>s}$$

divides $x_{\overline{s,i_s}} m$, and

$$m^j = m^j_{i=j} \cdot m^j_{i>j} \cdot m^j_{i<j} \cdot m^j_{>s}$$

divides $x_{\overline{t,j_t}} m$.

There is now a map $\ell : P \to [n]$ defined by

$$\ell(p) = \begin{cases} i_p & \text{for } p > s \\ \min(i_p, j_p) & \text{for not } (p > s) \end{cases}.$$

This is an isotone map as is easily checked. Its associated monomial is

$$m^\ell = m_{i=j} \cdot m^j_{i>j} \cdot m^i_{i<j} \cdot m^i_{>s}.$$

We will show that this divides m. Since the isotone map ℓ is $\leq$ the isotone map i, this will prove the theorem.

Claim 3. $m^j_{i>j}$ is relatively prime to $m^i_{i<j}$ and $m^i_{>s}$.

Proof. Let $x_{\overline{p,j_p}}$ be in $m^j_{i>j}$.

1. Suppose it equals the variable $x_{\overline{q,i_q}}$ in $m^i_{i<j}$. Then p and q are comparable since ϕ has left strict chain fibers. If $p < q$ then $j_p \geq i_q \geq i_p$, contradicting $i_p > j_p$. If $q < p$ then $i_q \geq j_p \geq j_q$ contradicting $i_q < j_q$.

2 Suppose $x_{\overline{p,j_p}}$ equals $x_{\overline{q,i_q}}$ in $m^i_{>s}$. Then p and q are comparable and so $p < q$ since $q > s$ and we do not have $p > s$. Then $j_p \geq i_q \geq i_p$ contradicting $i_p > j_p$. □

Let $abc = m^i_{i=j} \cdot m^i_{i<j} \cdot m^i_{>s}$ which divides $mx_{\overline{s,i_s}}$ and $ab' = m^j_{i=j} \cdot m^j_{i>j}$ which divides m since $x_{\overline{t,j_t}}$ is a factor of $m^j_{>s}$ since $t > s$. Now if the product of monomials abc divides the monomial n and ab' also divides n, and b' is relatively prime to bc, then the least common multiple $abb'c$ divides n. We thus see that the monomial associated to the isotone map ℓ

$$m^\ell = m_{i=j} \cdot m^j_{i>j} \cdot m^i_{i<j} \cdot m^i_{>s}$$

divides $mx_{\overline{s,i_s}}$. We need now only show that the variable $x_{\overline{s,i_s}}$ occurs to a power in the above product for m^ℓ less than or equal to that of its power in m.

Claim 4. $x_{\overline{s,i_s}}$ is not a factor of $m^j_{i>j}$ or $m^i_{i<j}$.

Proof. 1. Suppose $\overline{s,i_s} = \overline{p,i_p}$ where $i_p < j_p$ and not $p > s$. Since p and s are comparable (they are both in a fiber of ϕ), we have $p \leq s$. Since ϕ is isotone $i_p \leq i_s$ and since ϕ has left strict chain fibers $i_p \geq i_s$. Hence $i_p = i_s$. By (10) $j_s \leq i_s$ and so $j_p \leq j_s \leq i_s = i_p$. This contradicts $i_p < j_p$.

2. Suppose $\overline{s,i_s} = \overline{p,j_p}$ where $j_p < i_p$ and not $p > s$. Then again $p \leq s$ and $i_p \leq i_s \leq j_p$, giving a contradiction. □

If now $i_s > j_s$ then $x_{\overline{s,i_s}}$ is a factor in $m^i_{i>j}$ but by the above, not in $m^j_{i>j}$. Since m^ℓ is obtained from m^i by replacing $m^i_{i>j}$ with $m^j_{i>j}$, we see that m^ℓ contains a lower power of $x_{\overline{s,i_s}}$ than m^i and so m^ℓ divides m.

Claim 5. Suppose $i_s = j_s$. Then the power of $x_{\overline{s,i_s}}$ in $m^i_{>s}$ is less than or equal to its power in $m^j_{>s}$.

Proof. Suppose $\overline{s, i_s} = \overline{p, i_p}$ where $p > s$. We will show that then $i_p = j_p$. This will prove the claim.

The above implies $\overline{\overline{p, i_p}} = \overline{\overline{s, i_s}} = \overline{\overline{t, j_t}}$, so either $s < p < t$ or $s < t \leq p$. If the latter holds, then since ϕ has left strict chain fibers, $i_s \geq j_t \geq i_p$ and also $i_s \leq i_p$ by isotonicity, and so $i_s = i_p = j_t$. Thus

$$\overline{s, i_s} \leq \overline{t, j_t} \leq \overline{p, i_p}$$

and since the extremes are equal, all three are equal contradicting the assumption that the two first are unequal.

Hence $s < p < t$. By assumption on the fibre of ϕ we have $i_s \geq i_p$ and by isotonicity $i_s \leq i_p$ and so $i_s = i_p$. Also by (10) and isotonicity

$$i_s \geq j_t \geq j_p \geq j_s.$$

Since $i_s = j_s$ we get equalities everywhere and so $i_p = j_p$, as we wanted to prove. □

In case $i_s > j_s$ we have shown before Claim 5 that m^ℓ divides m. So suppose $i_s = j_s$. By the above two claims, the $x_{\overline{s,i_s}}$ in m^ℓ occurs only in $m_{i=j} \cdot m^i_{>s}$ and to a power less than or equal to that in $m_{i=j} \cdot m^j_{>s}$. Since m^j divides $mx_{\overline{t,j_t}}$ and $\overline{s, i_s} \neq \overline{t, j_t}$ the power of $\overline{s, i_s}$ in m^j is less than or equal to its power in m. Hence the power of $x_{\overline{s,i_s}}$ in m^ℓ is less or equal to its power in m and m^ℓ divides m.

□

Remark 8.2. Suppose P has a unique maximal element p. The above proof still holds if $\mathcal{J}$ in $\mathrm{Hom}(P, [n])$ is a poset ideal for the weaker partial order $\leq^w$ on $\mathrm{Hom}(P, [n])$ where the isotone maps $\phi \leq^w \psi$ if $\phi(q) \leq \psi(q)$ for $q < p$, and $\phi(p) = \psi(p)$.

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